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JOURNAL OF ALGEBRA 69, 455–466 (1981)

A Characterization of $M(22)$

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Communicated by W. Feit

Received July 19, 1977

0. INTRODUCTION

The finite simple group $M(22)$ was discovered by Fischer [3] in the context of his analysis of groups generated by a class of involutions of 3-transposition type. Classification of this group by the structure of the centralizer of an involution exists [11, 14]. Also $M(22)$ has been recently classified by possession of a standard component isomorphic to $O^+(8, 2)$ [13]. This paper proves

THEOREM. *Assume G is fusion-simple containing a group $E \cong E_{64}$ such that $C_G(E) = E$ and $N_G(E)/E \cong \mathrm{Sp}(6, 2)$. Then either $G = N_G(E)$ or else $N_G(E)$ is a split extension and $G \cong M(22)$.*

The method of proof considers separately the case $N_G(E)$ splits over E or not. In the non-split case we reduce to showing a Sylow 2-subgroup of G lies in $N_G(E)$ by studying a related 2-local subgroup. Then Goldschmidt's result [5] and the structure of $N_G(E)$ show E is normal in G.

If $N_G(E)$ does split over E then we show $G = N_G(E)$ iff a Sylow 2-subgroup of G lies in $N_G(E)$. If $G \not\supseteq N_G(E)$ then we determine approximately the centralizers of the 3 classes of involutions in G, using results of Parrott [14] and Goldschmidt [5]. Identification of G with $M(22)$ is accomplished by a result of Hunt [11].

All notation is standard with free usage of the "bar convention."

1. THE SYLOW 2-SUBGROUP OF $N_G(E)$ IN THE SPLIT EXTENSION CASE

The structure of a Sylow 2-subgroup of $N_G(E)$ has been determined by Dempwolff [2]. This structure is as follows:

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0021-8693/81/040455–12$02.00/0
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(a) Let $E = \langle v_1, v_2, v_3, v_4, v_5, v_6 \rangle$.

(b) Let $\{u_j \mid 1 \leqslant j \leqslant 9\}$ be a set of involutions with the following commutator identities:

$$[u_1, u_3] = u_2, \quad [u_1, u_7] = u_4, \quad [u_1, u_8] = u_5 u_6,$$
$$[u_3, u_4] = u_5, \quad [u_3, u_9] = u_7 u_8, \quad [u_2, u_7] = u_5,$$
$$[u_2, u_9] = u_4 u_6,$$

and

$$[v_1, u_1] = v_2, \quad [v_1, u_2] = v_3, \quad [v_1, u_4] = v_4,$$
$$[v_1, v_5] = v_5, \quad [v_1, u_6] = v_6, \quad [v_2, u_3] = v_3,$$
$$[v_2, u_7] = v_4, \quad [v_2, u_8] = v_5, \quad [v_2, u_5] = v_6,$$
$$[v_3, u_4] = v_6, \quad [v_3, u_9] = v_4, \quad [v_3, u_7] = v_5,$$
$$[v_4, u_2] = v_6, \quad [v_4, u_3] = v_5, \quad [v_5, u_1] = v_6.$$

with all unstated identities being trivial.

Then $\langle E, u_j \mid 1 \leqslant j \leqslant 9 \rangle$ is isomorphic to a Sylow 2-subgroup of $N_G(E)$ in the split extension case. Note $N_G(E)$ acts transitively on $E \backslash \{1\}$. Let $R = \langle u_6, u_5, u_4, u_2, u_1, v_6, v_5, v_4, v_3, v_2 \rangle$. Then $R \lhd C_{N(E)}(u_6)$ and $N_G(R) \subseteq C_G(v_6)$, since $R \cong Z_2 \times D_8 * D_8 * D_8 * D_8$ with $Z(R) = \langle u_6, v_6 \rangle$ and $R' = \langle v_6 \rangle$. Also we know [16] $N(R) \cap N(E)/R \cong Z_2 \times S_6$ with $v_1 R \in Z(N(R) \cap N(E)/R)$.

Some detailed information about $N(R) \cap N(E)$ will be used. In summary the major facts are:

(a) If $x \in N(R)$ and $\langle x \rangle$ fixes $v_1 R$ then $\langle x \rangle$ acts on E, the group generated by all involutions in $v_1 R$.

(b) If $Q \in \mathrm{Syl}_5(N(R) \cap N(E))$ then $C_R(Q) = Z(R)$. (Obvious since $R/\langle u_6 \rangle = [R/\langle u_6 \rangle, Q] * C_R\langle u_6 \rangle(Q)$.)

(c) There are 2 classes of elements of order 3 in $N(R) \cap N(E)$. If representatives are denoted by ρ and σ then $C_R(\sigma) = Z(R)$, while $C_R(\rho) = \langle u_6, u_5, v_2, u_1, v_5 \rangle$ and $\langle v_3^\rho \rangle = \langle v_3, v_3^\rho \rangle$. (This is taken directly from Yamaki's work on Sp(6, 2) [16].)

We keep this notation in Sections 2, 3, and 4.

2. The Structure of $N_G(R)$ in the Case $N_G(E)$ Splits over E

From Remark 1(a) it is obvious $C_G(R) = Z(R) \times O(C(R))$.

Let $\bar{N} = N(R)/RC(R)$. We observe $O_2(\bar{N}) \subseteq \langle \bar{v}_1 \rangle$ and for $\bar{x} \in \bar{N}$ of odd order we see $\langle \bar{x} \rangle$ acts trivially on $Z(R) \supseteq R' \supseteq 1$, so acts non-trivially on $R/\langle u_6 \rangle \cong D_8 * D_8 * D_8 * D_8$. Recall the outer automorphism group of $R/\langle u_6 \rangle$ is isomorphic to $O^+(8, 2)$ of order $2^{13}3^5 5^2 7$. By a result of Harris and Solomon [10] and the p-local structure of $O^+(8, 2)$ we conclude $O^2(\bar{N})$ must be isomorphic to A_8, or modulo a core $\bar{K}$ of order at most 3 is isomorphic to $P\ \mathrm{Sp}(4, 3)$ or A_6.

Suppose $\bar{M} = N(R)'/RC(R) \cong A_8$. Recall $u_6^{v_1} = u_6 v_6$ so all involutions in $N(R) \backslash N(R')$ do not commute with u_6. Since $[\langle u_3, u_7, u_8, u_9 \rangle u_6] = 1$ we conclude $\langle u_3, u_7, u_8, u_9 \rangle \subseteq N(R)'$. A computation reveals $C_{R/Z(R)}(u_7) \cong E_{16}$ while $C_{R/Z(R)}(u_8) \cong E_{2^6}$. Since elements of order 5 in $\bar{N}$ act without fixed points on $R/Z(R)$ we conclude $\bar{u}_8$ inverts no element of order 5 in $\bar{N}$. From the character table of A_8 we see non-central involutions do invert elements of order 5, so we conclude $\bar{u}_8$ is central.

Let $\bar{S} \in \mathrm{Syl}_2(\bar{M})$, $(\bar{u}_3, \bar{u}_8, \bar{u}_9) \subseteq \bar{S}$, and $Z(\bar{S}) = (\bar{u}_8)$. Let $\bar{F} = J(\bar{S}) \cong E_{16}$. From the structure of A_8 we know $\bar{u}_8 \in \bar{F}$. Since $[\bar{u}_3, \bar{u}_9] = \overline{u_7 u_8}$ we find $\langle \bar{u}_7, \bar{u}_8 \rangle \subseteq \bar{F}$. Let $\bar{p}$ be an element of order 3 in $N_{\bar{N}}(\bar{F})$ centralizing $\bar{u}_7$ (since $\bar{u}_7$ is non-central such an element exists). Then $\langle \bar{p} \rangle$ acts on the group generated by all involutions in $u_7 R$ which is $\langle u_7, C_R(u_7) \rangle$. Now $u_7 u_8 R$ cannot be fixed by pR or else $u_8 R$ is fixed by pR, a contradiction to the structure of $N_{\bar{N}}(\bar{F})$. Note the group generated by all involutions in $u_7 u_8 R$ is $\langle u_7 u_8, C_R(u_7 u_8) \rangle$. Since $\overline{u_7 u_8}$ is non-central and $C_R(u_7 u_8) = C_R(u_7)$ we conclude a Sylow 3-subgroup of $N_{\bar{N}}(\bar{F})$ must act faithfully on $C_R(u_7) = C_R(u_7 u_8) = C_R(u_7, u_8)$. Since $\langle u_9 \rangle$ centralizes $C_R(u_7)$ we conclude $\bar{u}_9 \in \bar{F}$.

Note $J_0 = \langle u_9, u_8, u_7, C_R(u_7) \rangle \cong E_{2^9}$. Let u be an involution so that $F = \langle u, u_9, u_8, u_7, R \rangle$. Since $m(R) = 6$ we conclude $m(F) = 9$ or 10. Suppose $m(F) = 9$. Since F contains at most 2 subgroups isomorphic to J_0 all 3-elements in $N_{\bar{N}}(\bar{F})$ will act on $\langle \bar{u}_7, \bar{u}_8, \bar{u}_9 \rangle$ which is not so. We conclude $J = \langle u, J_0 \rangle \cong E_{2^{10}}$.

We claim next R is uniquely determined in S. Let R_1 be any subgroup of S isomorphic to R. Since S/R is of type S_8 we know $m(S/R) = 4$. Hence $|R_1 \cap R| \geqslant 2^6$. Note $v_1 R$ and $v_1 u_8 R$ are representatives in $N(R)$ of all classes of involutions in $N(R) \backslash N(R)'$. A calculation in $N_S(E)$ now shows $R_1 \subseteq N(R)'$. Since $N(R)'/R \cong A_8$ our analysis of $J(S)$ implies now $|R_1 \cap R| \geqslant 2^7$.

Recall $u_8 R$ is central in $N(R)'/R$ while $u_7 R$ is a non-central involution. Also the set of all involutions in $u_8 R$ lies in $\langle u_8, C_R(u_8) \rangle$ and those in $u_7 R$ fall in $\langle u_7, C_R(u_7) \rangle$. Since $\langle u_7, u_8, R \rangle \subseteq N_S(E)$ a calculation reveals $[u_7, x] \in \langle v_6 \rangle$ implies $[u_7, x] = 1$, and a like result holds for u_8 as well. Note for $x \in C_R(u_7)$ we have $|C_R(u_7 x)| \leqslant 2^6$ and for $w \in C_R(u_8)$ we have $|C_R(u_8 w)| \leqslant 2^7$. Hence either $R_1 = R$ or $|R_1 \cap R| = 2^7$ and $R_1 \cap R = C_R(u_8)$. But $\langle u_8 w, C_R(u_8) \rangle = \langle u_8, C_R(u_8) \rangle \supseteq \langle u_8, u_6, u_5, u_4, v_6, v_5, v_4 \rangle \cong E_{2^7}$. Since $m(R) = 6$ this is impossible. We conclude R is unique in $N(R)$ so $S \in \mathrm{Syl}_2(G)$.

Now $J_0 \subseteq J$ so $J \subseteq C(J_0)$. From the structure of $N(R)$ we conclude J is a Sylow 2-subgroup of $C(J_0)$. Now $N(E) \cap N(J_0)/J_0 \cong E_8 L_3(2)$ must act on J. Let $M = (N(R) \cap N(J))\, C(J)$ and $\bar{M} = M/C(J)$. Now $\bar{M}$ is isomorphic to a split faithful extension of E_{16} by a Sylow 3-normalizer of A_8, qo [8, II.2.2] implies $H(J)/C(J)$ is of type A_{10}. Note $O_2(\bar{M}) = \langle \bar{u}_1, \bar{u}_2, \bar{v}_2, \bar{v}_3 \rangle$ and from $N_S(E)$ we see $\bar{v}_3$ is central in $\bar{M}$. But $C_{\bar{S}}(\bar{v}_1) = \langle \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{u}_3 \rangle$ and $C_{\bar{S}}(\bar{v}_1)' = \langle \bar{v}_3 \rangle$, so $v_1 J \not\sim v_3 J$ in $N(J)/C(J)$. This is a contradiction, so $N(R)/RC(R) \not\cong S_8$.

In summary we have shown

LEMMA 2.1. *If $N(E)$ splits over E then $R = \langle u_6, u_5, u_4, u_2, u_1, v_6, v_5, v_4, v_3, v_2 \rangle$ has the property $N(R)/RC(R)$ modulo a core of order at most* 3 *is isomorphic to $Z_2 \times S_6$ or* $\mathrm{Aut}(P\,\mathrm{Sp}(4, 3))$.

3. THE CASE $N_G(E)$ IS SPLIT AND CONTAINS A SYLOW 2-SUBGROUP OF $\mathbb{G}$

Let $S = \langle E, u_j \mid 1 \leqslant j \leqslant 9 \rangle$. The case in question corresponds to $N_G(R)/RC(R) \subseteq S_3 \times S_6$. Our method here is to determine $N(J(S))/C(J(S))$ and then show directly E is strongly closed in S with respect to G,

Note $J(S) = \langle u_9, u_8, u_7, u_6, u_5, u_4, v_6, v_5, v_4 \rangle \cong E_{2^9}$ and $N(J(S)) \cap N(E)/J(S) \cong E_8 \cdot L_3(2)$ with $\langle v_1, v_2, v_3, J(S) \rangle = O_2(N(J(S)) \cap N(E))$.

LEMMA 3.1. $N(J(S)) = (N(J(S)) \cap N(E)) \cdot C(J(S))$.

Proof. Let $\bar{N} = N(J(S))/C(J(S))$. Note $\bar{N}$ is of type A_8. Next observe that for $\bar{v}$ an involution in $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ we have $m(C_{J(S)}(v)) = 6$ while for $v \in S \backslash \langle v_1, v_2, v_3, J(S) \rangle$ we have $m(C_{J(S)}(v)) \leqslant 5$. Hence $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is strongly closed in $\bar{S}$ with respect to $\bar{N}$. Let $\bar{K} = O(\bar{N})$ and assume $\bar{K} \neq \bar{1}$. Factor $\bar{K}$ by $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$. Since any involution $\bar{v} \in \langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ stabilizes the chain $J(S) \supseteq [J(S), v] \supseteq 1$ and $[J(S), v] = \langle v_4, v_5, v_6 \rangle$ we find $\bar{K}$ also stabilizes this chain, so $\bar{K} = \bar{1}$. We conclude $\bar{K} = 1$. Applying a result of Gorenstein and Harada [6] we conclude $\bar{N} = \overline{N(J(S)) \cap N(E)}$.

LEMMA 3.2. *E is strongly closed in S with respect to G. In particular E is normal in G.*

Proof. Since $N(E)/E \cong \mathrm{Sp}(6, 2)$ we may apply Yamaki's work [16] to conclude the representatives of all conjugacy classes of involutions in G are located in E, $u_6 E$, $u_5 E$, $u_6 u_5 E$, and $u_6 u_7 E$. Since $N(E)$ acts transitively on $E \backslash 1$ we will simply consider fusion involving $v_6 \in Z(S)$.

Consider u_6u_7E. Note any involution i in u_6u_7E has the property $J(S) \subseteq C_S(i)$. By Lemma 3.1 we immediately conclude no fusion occurs between u_6u_7E and E. Next consider u_6E. From the action of $C(u_6) \cap N(E)$ on $C_E(u_6) = \langle v_2, v_3, v_4, v_5, v_6\rangle$ we note it suffices to consider u_6v_5 (or $u_6v_5v_6 = (u_6u_5)^{v_1}$). But $J(S) \subseteq C_S(u_6v_5)$ so again no fusion occurs between u_6E and E itself. Using the element ρ of order 3 mentioned in Remark 1(c) we may simply consider fusion between u_5 and u_5v_4 or u_5u_6 and $u_5u_6v_4$ and v_6. A final application of Lemma 3.1 now shows E is strongly closed in S with respect to G. By a result of Goldschmidt [5] and the structure of $N(E)$ we see $E \lhd G$.

4. The Case $N(E)$ Splits over E But Does Not Contain a Sylow 2-Subgroup of G

In this situation we have by Lemma 2.1 $N(R)'/RC(R) \cong Z_3 \times P\,\mathrm{Sp}(4, 3)$. We show first $N(R)$ does contain a Sylow 2-subgroup of G. Let $S \in \mathrm{Syl}_2(N(R))$ containing $\langle E, u_j \mid 1 \leqslant j \leqslant 9\rangle$. We show then $J(S) \cong E_{2^{10}}$ and $N(J(S))/C(J(S)) \cong M_{22}$. The approximate structure of the centralizer of every involution is determined. In particular a result of Hunt [11] applies since $C_G(u_6) \cong \widehat{U_6}(2)$, the central extension (perfect) of $U_6(2)$ by Z_2.

Lemma 4.1. *R is weakly closed in S with respect to $C_G(v_6)$. In particular $S \in \mathrm{Syl}_2(G)$.*

Proof. Let $H = C_G(v_6)$ and $\bar{H} = H/\langle v_6\rangle$. It is enough to show $\bar{R}$ is unique in $\overline{N(R)}$ with respect to $\bar{H}$. Assume $\bar{R}_1$ is an $\bar{H}$-conjugate of $\bar{R}$ contained in $\overline{N(R)}$. Now any involution in $\overline{N(R)}\backslash\overline{N(R)'}$ is $\overline{N(R)'}$-conjugate to $\bar{v}_1$ or $\overline{u_8v_1}$. From the structure of $N(R) \cap N(E)$ we conclude $\bar{R}_1 \subseteq \overline{N(R)'}$. Since $\overline{N(R)'}/\bar{R}$ is of type A_8 we see $\bar{R}_1/\bar{R}_1 \cap \bar{R}$ has 2-rank at most 4. Suppose $\bar{R}_1 \cap \bar{R} \cong E_{2^5}$. Then $[u_3, u_9] = u_7u_8$ implies $\overline{u_7u_8} \in \bar{R}_1$. But $C_R(u_7u_8) = \langle u_6, u_5, u_4, v_6, v_5, v_4\rangle \cong E_{64}$, so $m(R_1) \geqslant 7$. This is a contradiction, so $|\bar{R}_1 \cap \bar{R}| \geqslant 2^6$.

From the character table of $PS_p(4, 3)$ we observe only noncentral involutions invert elements of order 5. Since 5-elements in $\overline{N(R)'}/C(\bar{R})$ act *fpf* on $R/Z(R)$, we conclude $\bar{R}_1/\bar{R} \cap \bar{R}_1$ contains no non-central involutions of $\overline{N(R)'}/C(\bar{R})$. (Non-central involutions in $\overline{N(R)'}/C(\bar{R})$ can fix exactly half of $\bar{R}$.) But no four-group in $P\,\mathrm{Sp}(4, 3)$ contains 3 central involutions, so $\bar{R} \cap \bar{R}_1$ has order 2^8. Since $N(R)' \cap N(E)/R$ contains representatives of all conjugacy classes of involutions in $N(R)'/R$ and $C_{\bar{R}}(\bar{u}_8) = \langle \bar{u}_6, \bar{u}_5, \bar{u}_4, \bar{u}_1, \bar{v}_5, \bar{v}_4, \bar{v}_3\rangle \cong E_{2^7}$ we obtain a contradiction. The second assertion is immediate.

Lemma 4.2. *$J(S) \cong E_{2^{10}}$ and $N(J(S))/C(J(S)) \cong M_{22}$, the Mathieu group on 22 letters. Further the action of $N(J(S))$ on $J(S)$ is uniquely determined and irreducible. Also $N(R)/RC(R) \cong \mathrm{Aut}(P\,\mathrm{Sp}(4, 3))$.*

Proof. Let $\bar{N}=N(R)'/R\cdot C(R)$. In $\overline{S\cap N(R)'}$ let $\bar{P}\cong E_{16}$. Since $\overline{S\cap N(R)'}$ is of type A_8, $\bar{P}$ is uniquely determined. Since $[u_3,u_9]=u_7u_8$ and $\langle\bar{u}_3,\bar{u}_9\rangle\subseteq\overline{S\cap N(R)'}$ we conclude $\overline{u_7u_8}\in\bar{P}$. Note $C_R(u_7u_8)=\langle u_6, u_5, u_4, v_6, v_5, v_4\rangle$. Now $N_{\bar{N}}(\bar{P})\cong E_{16}\cdot A_5$ will act on the center of $\bar{P}/\langle\bar{u}_6,\bar{v}_6\rangle$. Since a Sylow 5-subgroup of $N_{\bar{N}}(\bar{P})$ acts fixed point feely on $R/\langle u_6,v_6\rangle$ we conclude the center of $\bar{P}/\langle\bar{u}_6,\bar{v}_6\rangle$ is $C_R(u_7u_8)$. Since $\langle u_7,u_8,u_9\rangle$ centralizes $\overline{C_R(u_7u_8)}$ we conclude $\langle\bar{u}_7,\bar{u}_8,\bar{u}_9\rangle\subseteq\bar{P}$.

Let u be an involution in $P\backslash\langle u_7,u_8,u_9,R\rangle$. If we cannot choose u to centralize $\langle u_7,u_8,u_9,C_R(u_7u_8)\rangle$ then we contradict the action of 5-elements in $\bar{N}$. Hence $J=\langle u,u_7,u_8,u_9,C_R(u_7u_8)\rangle\cong E_{2^{10}}$. Since the 2-rank of R is 6 and all involutions in $\overline{N(R)}\backslash\bar{N}$ centralize at most an E_{32}-subgroup of R we see J is uniquely determined in S. Hence $J(S)\cong E_{2^{10}}$.

Let $M=N(J(S))$ and $\bar{M}=N(J(S))/C(J(S))$. Now $\bar{S}\in\mathrm{Syl}_2(\bar{M})$ and $\bar{S}$ is a Sylow 2-subgroup of $\bar{B}=\overline{N(R)\cap N(J(S))}$. Since $\bar{B}/O_2(\bar{B})$ is isomorphic to a subgroup of $N_{A_8}(\langle 123\rangle)$ a result of Gorenstein and Harada [8, II.2.2] yields $\bar{S}$ is of type A_{10} or $\hat{A}_8$, as $\bar{B}$ acts intransitively on $O_2(\bar{B})$ or not. If a Sylow 3-subgroup of $\bar{B}$ is of order 3^2 then $\bar{B}$ is of type $\hat{A}_8$. Suppose then $\bar{B}/O_2(\bar{B})\cong S_5$ and the action is intransitive. From $\overline{N_S(E)}$ we see $Z(\bar{S})=\langle\bar{v}_3\rangle$, so there exists an element $\bar{\tau}\in\bar{B}$ of order 3 such that $[\bar{\tau},\bar{v}_3]=1$. Observe $D=C_{J(S)}(v_3)\cap R=\langle u_6,u_5,v_6,v_5,v_4\rangle$ admits $\langle\bar{\tau}\rangle$ non-trivially. But $v_6\in C_B(\bar{\tau})$ so $C_B(\bar{\tau})\cong E_8$. Now $\langle u,u_2,v_2,v_3\rangle$ acts non-trivially on $C_B(\bar{\tau})$, a contradiction. We conclude in both cases $\bar{S}$ is of type $\hat{A}_8$.

Let $T=N_S(E)$. We know $J(T)\subseteq J(S)$ so $N(J(T))\cap N(E)$ acts on a Sylow 2-subgroup of $C(J(T))$. Computing we see a Sylow 2-subgroup of $C(J(T))$ is $J(S)$. In particular $\bar{v}_1\in O^2(\bar{M})$. Since the 2'-share of $GL(10,2)$ is $3^6\cdot 5^2\cdot 7^3\cdot 31^2\cdot 73\cdot 127$ we know $\bar{M}/O(\bar{M})\not\cong M_{23}$. Hence $\bar{B}/O_2(\bar{B})\cong S_5$, since only M_{22} and M_{23} are fusion-simple of type $\hat{A}_8$ and involve $E_{16}S_5$. In particular $N(R)/R\cong\mathrm{Aut}(P\,\mathrm{Sp}(4,3))$.

Since $O(\bar{M})\subseteq Z(\bar{M})$ the $A\times B$-lemma shows $O(\bar{M})=\bar{1}$. Since $11\nmid|GL(9,2)|$ we also find the action is irreducible.

Finally we show the action of $\bar{M}$ on $J(S)$ is unique. From the decomposition matrix of M_{22} for the prime 2 [12] we know M_{22} has two inequivalent absolutely irreducible 2-modular representations of degree 10. One extension of $E_{2^{10}}$ by M_{22} occurs in Conway's group 0.2 with orbits 77, 330, and 616 [15]. The other extension of $E_{2^{10}}$ by M_{22} occurs in $M(22)$ with orbits 22, 231, and 770. Now $E_{16}\cdot S_5$ is a maximal subgroup of M_{22} so the action of $\bar{M}$ is uniquely determined.

LEMMA 4.3. *G has exactly 3 classes of involutions with representatives v_6, u_6, and u_6v_5. Further $C_G(u_6)\not\subseteq C_G(v_6)$.*

Proof. Note Lemma 4.1 implies $\langle v_6\rangle=R'$ char R char $C_S(u_6)$, so $v_6\not\sim u_6$. Since $N(J(S))$ controls fusion in $J(S)$ by Burnside's theorem we see we have 3 G-classes of involutions in $J(S)$ and the representatives may be chosen to lie in $Z_2(N_S(E)) = \langle u_6, v_5, v_6\rangle$. Indeed any such representative has a centralizer with 2-share of order 2^{16}, so must lie in R. Then from $|N_S(E)| = 2^{15}$ we have all representatives must lie in $Z_2(N_S(E))$. Since $u_6 \sim u_6 v_6$ and $v_6 \sim v_5 v_6 \sim v_5$ in $N_G(E)$ the representatives may be assumed to be u_6, v_6, and $u_6 v_5$. From the structure of $N_G(E)$ and $N_G(R)$ we see G contains exactly 3 classes of involutions. Finally no point in $J(S)$ has stabilizer in $\bar{M}$ isomorphic to $E_{16} \cdot A_5$. From the structure of M_{22} the stabilizer in $\bar{M}$ of u_6 can be isomorphic to $E_{16} \cdot A_5$ or $PSL(3,4)$. Hence it is isomorphic to $PSL(3,4)$, so $C_G(u_6) \not\subseteq C_G(v_6)$.

LEMMA 4.4. $C_G(v_6) = N_G(R) \cdot O(C_G(v_6))$.

Proof. Let $H = C_G(v_6)$ and $\bar{H} = H/\langle v_6\rangle$. We shall show first $\bar{R}$ is strongly closed in $\bar{S}$ with respect to $\bar{H}$. Consider the set

$$\mathscr{S} = \{\bar{B}_0 \subseteq \bar{S} \mid \bar{B}_0 \text{ conjugate to a subgroup of } \bar{R}, \bar{B}_0 \not\subseteq \bar{R}\}.$$

Let us ssume $\mathscr{S}$ is non-empty. Choose $\bar{B} \in \mathscr{S}$ with $r = m(\bar{B}/\bar{B} \cap \bar{R})$ as large as possible. We know $r \leqslant 4$ since $N(R)/R \cdot C(R) \cong \mathrm{Aut}(P\,\mathrm{SO}(4,3))$.

Let a word $\bar{\omega} \in \bar{B}$ non-trivially $\bar{v}_1$. Then $m([\bar{R}, \bar{\omega}]) \geqslant 4$ and if $r = 4$ then there is a word $\bar{\omega}_0 \in \bar{B}$ such that $m([\bar{R}, \bar{\omega}_0]) \geqslant 5$.

Next assume $\bar{B} \subseteq \overline{N(R)'}$. From the proof of Lemma 4.1 we know non-central involutions in $N(R)'/R$ centralize exactly half of $R/Z(R)$ and central involutions centralize a 6-dimensional space in $R/Z(R)$. Hence if $\langle \bar{t}\rangle = \bar{B}$ then $m([\bar{R}, \bar{t}]) \geqslant 2$. By Lemma 4.3 we know $r \neq 4$. Since no four-group in $P\,\mathrm{Sp}(4,3)$ contains 3 central involutions we see that if $r = 2$ then $\bar{B}$ contains an element $\bar{t} \in \bar{B}$ such that $m([\bar{R}, \bar{t}]) \geqslant 3$. Finally suppose $r = 3$. Since $\bar{B} \cap \overline{J(S)} \neq \bar{1}$ we contradict Lemma 4.3.

In summary we have shown that $\bar{B}$ contains an involution such that $m([\bar{R}, \bar{t}]) \geqslant 1 + r$. Hence the set $\mathscr{S}$ is in fact empty. Corollary 4 of [5] states if the set $\mathscr{S}$ is empty then $\bar{R}$ is strongly closed in $\bar{S}$ with respect to $\bar{H}$. By Corollary 4 of [5] we conclude $\bar{R}$ is strongly closed in $\bar{S}$ with respect to $\bar{H}$. By the structure of $N(R)$ we conclude $\bar{R} \cdot O(\bar{H}) \lhd \bar{H}$. Hence $H = N_G(R) \cdot O(H)$.

LEMMA 4.5. $C_G(u_6)/O(C_G(u_6)) \cong \hat{U_6}(2)$.

Proof. Let $M = C_G(u_6)$ and $\bar{M} = M/O(M)$. From Lemma 4.4 $C_{\bar{M}}(\bar{v}_6) = \bar{K}$ has the property $O_2(\bar{K}) \cong D_8 * D_8 * D_8 * D_8$ and $\bar{K}/O_2(\bar{K}) \cong P\,\mathrm{Sp}(4,3)$ with $\bar{Q} \in \mathrm{Syl}_5(\bar{K})$ such that $C_{O_2(\bar{K})}(\bar{Q}) = \langle \bar{v}_6\rangle$. Further Lemma 4.3 implies $\bar{v}_6$ is not isolated, so by a result of Parrot [14] $\bar{K} \cong U_6(2)$. From $N_S(E)$ we conclude $\bar{M} \cong \hat{U_6}(2)$.

LEMMA 4.6. *$C_G(u_6v_5)/O(C_G(u_6v_5))$ is 2-generated.*

Proof. Let $N=N(J(S))$ and $\bar{N}=N/O(N)$. From Lemmas 4.2 and 4.3 we have $C_{\bar{N}}(\overline{u_6v_5})/\overline{J(S)} \cong E_{16}P$ with $\omega \in C_S(u_6v_5)$ an involution such that $\langle\bar{\omega}, \bar{u}_3, \bar{u}_2, \bar{v}_3, J(\bar{S})\rangle = O_2(C_{\bar{N}}(\overline{u_6v_5}))$ and $C_{\bar{N}}(\overline{u_6v_5})/O_2(C_{\bar{N}}(\overline{u_6v_5}))$ isomorphic to a Sylow 3-normalizer in A_6. Note $D_0=[\langle u_3, v_3, u_2\rangle, J(S)]=\langle u_7u_8, u_4u_6, u_5v_6, v_5, v_4\rangle$. Since $\overline{J(S)}/\bar{D}_0$ admits non-trivial A_6 action we conclude $D_0 = O_2(C_{\bar{N}}(\overline{u_6v_5}))'$. Hence $D=\langle D_0, u_6v_5\rangle$ is an abelian normal subgroup of $C_S(u_6v_5) \in \mathrm{Syl}_2(C_G(u_6v_5))$. We also note $D \subseteq J(S)$ implies D is weakly closed in $C_S(u_6v_5)$ with respect to $C_G(u_6v_5)$.

We wish to show now D is strongly closed in $C_S(u_6v_5)$ with respect to $C_G(u_6v_5)$. Suppose $A=\langle u_2, u_3, v_3, \omega\rangle$ is conjugate to a subgroup of D. Then $C(A) \cap C_S(u_6v_5) = \langle A, v_6, v_5, u_6\rangle$ is conjugate to a subgroup of $J(S) = C(D_0) \cap C_S(u_6v_5)$. Since $N(J(S))$ controls fusion in $J(S)$ we may assume this conjugation takes place in $C(u_6v_5) \cap C(v_6)$. But $C(v_6)=N(R)\cdot O(C(v_6))$ by Lemma 4.4 and $D/D\cap R$ has order 2, a contradiction to $A/A\cap R$ has order 4. A similar argument shows $\langle u_2, u_3, v_3\rangle$ is not conjugate to any subgroup of D under the action of $C_G(u_6v_5)$.

From the structure of $C_{\bar{N}}(\overline{u_6v_5})$ we see any four-group in $C_S(u_6v_5)\backslash J(S)$ contains an involution t that has the property $m([D,t]) \geqslant 3$. Finally one checks that any involution $t \in C_S(u_6v_5)\backslash J(S)$ has the property $m([D,t]) \geqslant 2$. By Corollary 4 of [5] we conclude D is strongly closed in $C_S(u_6v_5)$ with respect to $C_G(u_6v_5)$. Since $D \subseteq J(S)$ the structure of $N(J(S))$ and [5] implies $D\cdot O(C_G(u_6v_5)) \lhd C_G(u_6v_5)$ as desired.

LEMMA 4.7. $G \cong M(22)$.

Proof. Since $m_2(G)=10$ we know G is connected. Since $U_6(2)$ is 2-generated Theorem C of [9] implies $O(C_G(u_6))=1$. Hence $C_G(u_6) \cong \hat{U_6}(2)$. By a result of Hunt [11] we conclude $G \cong M(22)$.

5. THE CASE $N_G(E)$ IS NON-SPLIT

In [2] Dempwolff shows a Sylow 2-subgroup T of a non-split extension of E by $N_G(E)$ with $E \cong E_{64}$ and $N_G(E)/E \cong \mathrm{Sp}(6,2)$ has generators $\langle E, u_j \mid 1 \leqslant j \leqslant 9\rangle$ with u_1, u_2, u_3, u_4, u_5, and u_7 involutions and $u_6^2=v_6$, $u_8^2=v_5$, and $u_9^2=v_4$. The commutator identities are the same as in the split extension case except:

$$[u_1,u_3]=u_2v_3v_6, \qquad [u_1,u_7]=u_4v_6, \qquad [u_1,u_8]=u_6u_5v_5v_6,$$
$$[u_2,u_7]=u_5v_6, \qquad [u_2,u_9]=u_6u_4v_4, \qquad [u_3,u_9]=u_8u_7v_5v_4,$$
$$[u_3,u_4]=u_5v_5.$$

Further we observe T has the following properties:

(a) $Z(T) = \langle v_6 \rangle$.

(b) If $T_0 \subseteq T$ is of order 2^{11} and $Z(T_0)$ has order at least 8 then either $u_5 \in Z(T_0)$ or $u_5 v_4 \in Z(T_0)$.

(c) $J(T) = \langle u_9, u_8, u_7, u_6, u_5, u_4 \rangle \cong Z_4 \times Z_4 \times Z_4 \times E_8$ with $\Omega_1(J(T)) = \langle v_6, v_5, v_4, u_7, u_5, u_4 \rangle$ and $\mho'(J(T)) = \langle v_4, v_5, v_6 \rangle$.

Our next goal is to show $N(E)$ must contain a Sylow 2-subgroup of G. To do this we again consider the structure of $R = \langle u_6, u_5, u_4, u_2, u_1, v_6, v_5, v_4, v_3, v_2 \rangle \cong Z_4 * D_8 * D_8 * D_8 * D_8$. As in the split extension case we have $N(R) \cap N(E)/R \cong Z_2 \times S_6$ with a Sylow 5-subgroup of $N(R) \cap N(E)$ acting fixed point freely on $R/Z(R)$. Application of [10] gives the following possibilities for $N(R)/RC(R)$:

(a) S_8.

(b) $\mathrm{Aut}(P\,\mathrm{Sp}(4, 3))$ modulo a core of order at most 3.

(c) $\mathrm{Sp}(4, 4) \cdot \langle f \rangle$ with $\langle f \rangle \cong \mathrm{Gal}(G(4))$.

(d) A normal subgroup $\cong A_6 \times A_6$.

(e) $v_1 RC(R) \in Z^*(N(R)/RC(R))$.

Suppose $N(R)/RC(R) \cong S_8$. Again we observe all involutions outside $N(R)'$ do not centralize $Z(R)$ so $\langle u_8, u_3, u_9, u_7 \rangle \subseteq N(R)'$. Also as in the split extension case we conclude that in $\bar{N} = N(R)'/RC(R)\, \bar{u}_8$ is central. Let $\bar{S} \in \mathrm{Syl}_2(\bar{N})$ wth $\langle \bar{u}_8 \rangle = Z(\bar{S})$ and set $\bar{F} = J(\bar{S}) \leqq_{16}$. As in the split case we find $\langle \bar{u}_7, \bar{u}_8 \rangle \subseteq \bar{F}$ since $[\bar{u}_3, \bar{u}_9] = \overline{u_7 u_8}$. Since a Sylow 3-subgroup of $N_{\bar{N}}(\bar{F})$ must act on $\langle u_7, u_8, C_R(u_7, u_8) \rangle$ and $[u_9, C_R(u_7, u_8)] = 1$ we find $\bar{u}_9 \in \bar{F}$. Now $J_0 = \langle u_7, u_8, u_9, C_R(u_7) \rangle \cong Z_4 \times Z_4 \times Z_4 \times E_8$ so if we choose an involution $\bar{u} \in \bar{F}$ such that $\langle \bar{u}, \bar{u}_9, \bar{u}_8, \bar{u}_7 \rangle \cong E_{16}$ then $J = \langle u, J_0 \rangle$ is isomorphic to either $Z_4 \times Z_4 \times Z_4 \times E_{16}$ or $Z_4 \times Z_4 \times Z_4 \times Z_4 \times E_8$. Note $N(J_0) \cap N(E)$ must act non-trivially on J as well. Hence $N(J)/C(J)$ must involve A_8, contradicting the action of $N(J_0)$ on J, or a 3-element in $N(R)/RC(R)$ acts trivially on R, a contradiction. In either case we find $N(R)/RC(R) \not\cong S_8$. Suppose then case (b) occurs. Arguing as in Lemma 4.1 we see $N(R)$ must contain a Sylow 2-subgroup S of G. But now the proof of Lemma 4.2 implies $J(S)$ is isomorphic to either $Z_4 \times Z_4 \times Z_4 \times E_{16}$ or $Z_4 \times Z_4 \times Z_4 \times Z_4 \times E_8$ and $N(R) \cap N(J(S))$ admits faithful action by A_5. This is impossible since $u_6 \in J(S)$.

Suppose we are in case (c) now. Using the element ρ of order 3 mentioned in Remark 1(c) we check $[u_8, \rho] = 1$. If $\bar{N} = N_G(R)/R \cdot C_G(R)$ then $\bar{u}_8$ is central in $\bar{N}'$. Now a Sylow 2-subgroup of $\mathrm{Sp}(4, 4)$ has class 2 so $\langle \bar{u}_7, \bar{u}_8 \rangle = Z(\bar{S})$. Note $C_R(u_7, u_8) = \langle u_6, u_5, u_4, v_5, v_4 \rangle$ with $C_R(u_8) = \langle u_6, u_5, u_4, v_5, v_4, u_2, v_3 \rangle$ and $Z(C_R(u_8)) = \langle u_6, u_5, v_5 \rangle$. Hence $\bar{N}'$ contains a goup $\bar{A} \cong A_5$ with

$[\bar{u}_8, \bar{A}] = \bar{1}$ and $\bar{A}$ acting trivially on $R \supseteq C_R(u_8) \supseteq C_R(u_7, u_8) \supseteq \langle u_6, u_5, v_5 \rangle \supseteq 1$. This is impossible.

Finally we consider case (d). Let $A_0 \supseteq R \cdot C(R)$ be a subgroup of $N(R)$ such that $\bar{A}_0 = A_0/RC(R) \cong A_6 \times A_6$. Let $\bar{t}$ be an involution in a direct factor of $\bar{A}_0$ acting on $J(T)$. Now $\bar{t}$ acts on $B = \Omega_1(J(T)) \cap R = \langle v_6, v_5, v_4, u_5, u_4 \rangle$. If $\langle \bar{t} \rangle$ centralizes B then $\langle \bar{t}^{v_1} \rangle$ also centralizes B, since $B^{v_1} = B$. Let $\bar{C} = C_{\bar{A}_0}(\bar{t})^{(\infty)} \cong A_6$. Since $\bar{t}^{v_1} \in \bar{C}$ we have $\bar{C}$ centralizes B. Hence $\bar{A}_0$ centralizes B, a contradiction.

Suppose $\langle \bar{t} \rangle$ acts non-trivially on B. If $\langle \bar{t} \rangle$ centralizes $\langle v_6, v_5, v_4 \rangle$ then $\bar{t}^{v_1} = \bar{t}$ since the stability group must be abelian. This is not so, so $\langle \bar{t} \rangle$ must act non-trivially on $\langle v_6, v_5, v_4 \rangle$. Then for some i, j, k we have $[tu_1^i u_2^j u_3^k, \langle v_6, v_5, v_4 \rangle] = 1$. Again $[v_1, tu_1^i u_2^j u_3^k] = 1$. So $\overline{tu_3}^{v_1} = \overline{tu_3}$ showing $(tR)^{v_1} = tR$ a contradiction. We conclude case (d) does not occur.

We summarize this set of observations as

LEMMA 5.1. *If $N_G(E)$ is a non-split extension of E by* Sp(6, 2) *then $N(R)/R \cdot C(R)$ contains an isolated involution $v_1 RC(R)$.*

Next note $\langle v_1 R \rangle$ contains exactly two type E_{64} subgroups, namely, E and $F = \langle u_6 v_1, v_2, v_3, v_4, v_5, v_6 \rangle$. Let $x \in C(R) \backslash Z(R)$ such that $v_1^x = u_6 v_1$. Then $v_1^{x^2} = (u_6 v_1)^x = u_6^x u_6 v_1$. Hence if $x^2 \in \langle E, u_j \mid 1 \leqslant j \leqslant 9 \rangle$ then $u_6^x = u_6 v_6$, a contradiction. From the structure of $N(E)$ we conclude $C(R) = Z(R) \times O(C(R))$. Hence if there is a 2-element $x \in N(R)$ that fuses E to F then $x \notin C(R)$. In particular $N(R)/RC(R) \cong Z_2 \times Z_2 \times S_6$. Since 5-elements in $N(R)/RC(R)$ act fixed point freely on $R/Z(R)$ we find $[\langle x \rangle, E \cap F] = 1$.

Now consider $N = N(J(T))$ and $\bar{N} = N(J(T))/C(J(T))$. Note $\langle x \rangle$ acts non-trivially on $\langle u_6 \rangle$ so $\bar{x} \neq \bar{1}$. If $\mathscr{C}: J(T) \supseteq \Omega_1(J(T)) \supseteq \mho^1(J(T)) \supseteq 1$ then $\bar{x}$ must act on $\mathscr{C}$. Recall $\overline{N \cap N(E)} \cong E_8 \cdot L_3(2)$ with $O_2(\overline{N \cap N(E)}) = \langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ stabilizing $\mathscr{C}$. Hence $\bar{x}$ centralizes $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$. If $\bar{P} \in \mathrm{Syl}_7(\overline{N \cap N(E)})$ then we may choose $\bar{x}_0$ to centralize $\bar{P}$ and $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$, where $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{x} \rangle = \langle \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{x}_0 \rangle$. But $\bar{P}$ acts fixed point freely on $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ a contradiction to the action of $\bar{x}_0$ on $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$. We conclude $N(E)$ contains a Sylow 2-subgroup of G. Since each factor of $\mathscr{C}$ is elementary of order 8 we also conclude the $2'$-share of $\bar{N}$ is that of $\overline{N \cap N(E)}$ which is 3.7. This in summary is

LEMMA 5.2. *$N(E)$ contains a Sylow 2-subgroup of G. Further $N(J(T)) = (N(J(T)) \cap N(E)) \cdot C(J(T))$.*

We are done now.

LEMMA 5.3. *E is normal in G.*

Proof. We shall prove E is strongly closed in T with respect to G. The result will follow upon application of Goldsmidt's result [5] and the structure of $N(E)$. Since $N(E)$ acts transitively on $E^{\#}$ it suffices to consider fusion patterns involving v_6. From Yamaki's work [16] we know the representatives of all conjugacy classes of involutions are contained in u_6E, u_5E, u_5u_6E, u_6u_7E, and E itself.

Using the element ρ of order 3 mentioned in Remark 1(c) and the action of T on u_5E we see it will suffice to consider fusion between u_5 or u_5v_4 and v_6. Since $C_T(u_5)$ and $C_T(u_5v_4)$ contain $J(T)$, Lemma 5.2 implies no fusion occurs between u_5E and E.

Next examine u_5u_6E. Again using ρ and T we see it is enough to consider fusion between $u_5u_6v_2$ and v_6. Now $C_T(u_6u_5v_2)$ has order 2^{11} and $L_3(C_T(u_6u_5v_2)) = \langle v_6, v_4\rangle$, $L_4(C_T(u_6u_5v_2)) = \langle v_6\rangle$ and $Z(C_T(u_6u_5v_2)) = \langle v_6, u_5u_6v_2\rangle$. Suppose there exists $T_1 \subseteq C(u_6u_5v_2)$ with $T_1 \supseteq C_T(u_6u_5v_2)$ and $[T_1: C_T(u_6u_5v_2)] = 2$. Then within T_1 there is a subgroup T_0 of order $2''$ with $Z(T_0) = \langle u_6u_5v_2, v_6, v_4\rangle$. Since all involutions in $Z(T_0)$ are conjugate in G, we contradict $u_5 \nsim v_6$ using Remark 1(b). We conclude no fusion occurs between u_5u_6E and E.

Consider next u_6E. Since all involutions in u_6E are fused in T we will consider $C_T(u_6v_1)$. Let $x \in C(u_6v_1)\backslash C_T(u_6v_1)$ be a 2-element acting on $C_T(u_6v_1)$. We check $L_4(C_T(u_6v_1)) = \langle v_5\rangle$ so $[v_5, x] = 1$. If $[x_1, v_6] = 1$ then $A = \langle x, C_T(u_6v_1)\rangle$ has order at least $2''$ and $Z(A) \supseteq \langle v_6, v_5, u_6v_1\rangle$. Again this implies $v_6 \sim u_5$ a contradiction. Hence $\langle x\rangle$ acts non-trivially on $Z(C_T(u_6v_1)) = \langle v_6, v_5, u_6v_1\rangle$. We observe $\langle u_6, u_5\rangle$ act non-trivially on $Z(C_T(u_6v_1))$ and $\langle u_6, u_5, C_T(u_6v_1)\rangle \supseteq J(T)$. This contradicts Lemma 5.2. We conclude u_6E has no conjugate of v_6.

Finally consider u_6u_7E. Again there is one class of involutions already under T so we will simply examine $C_T(u_6u_7v_1)$. We check $|C_T(u_6u_7v_1)| = 2^{10}$ and $L_3(C_T(u_6u_7v_1)) = \langle v_5, v_4\rangle$, $L_4(C_T(u_6u_7v_1)) = \langle v_5\rangle$, and $Z(C_T(u_6u_7v_1)) = \langle v_6, v_5, u_6u_7v_1\rangle$. If $u_6u_7v_1 \sim v_6$ then we choose a 2-element $x \in C_G(u_6u_7v_1)\backslash C_T(u_6u_7v_1)$ normalizing $C_T(u_6u_7v_1)$. As in the previous case we have $[x, v_6] \neq 1$. Let $A = \langle v_6, v_5, v_4, u_6u_7v_1\rangle$. Note $\bar{N} = N(A)/C(A)$ is isomorphic to a subgroup of A_8 and $\langle \bar{u}_6, \bar{u}_5, \bar{u}_4, \bar{u}_3, \bar{x}\rangle \subseteq \bar{N}$. If the 2-share of $\bar{N}$ is bigger than 2^4 then $u_6u_7v_1 \sim v_6$ within $N(J(T))$ since $\langle u_6, u_5, u_4, C_T(u_6u_7v_1)\rangle \supseteq J(T)$. This contradicts Lemma 5.1. Note $\langle \bar{u}_6, \bar{u}_5, \bar{u}_4, \bar{u}_3\rangle \cong Z_2 \times D_8$ implying $\bar{N}$ is a subgroup of S_6 involving $Z_2 \times S_4$. Let $\bar{M} \subseteq \bar{N}$ be isomorphic to $Z_2 \times S_4$ containing $\langle \bar{u}_6, \bar{u}_5, \bar{u}_4, \bar{u}_3\rangle$. Since $u_6^2 = v_6$ we have an element $\bar{\tau} \in \bar{M}$ of order 3 such that $[\bar{\tau}, \bar{u}_6] = \bar{1}$. Hence $[\bar{\tau}, v_6] = 1$ since $[A, \langle u_6\rangle] = \langle v_6\rangle$. If $\langle \bar{u}_6, \bar{u}_5, \bar{u}_4\rangle = O_2(\bar{M})$ then again we contradict Lemma 5.1. Hence we may assume $\langle \bar{u}_6, \bar{u}_5, \bar{u}_3\rangle = O_2(\bar{M})$. (Note $[\bar{u}_4, \bar{u}_3] = \bar{u}_5$.) Then $[v_5, \bar{\tau}] = 1$. But u_4 stabilizes $A \supseteq \langle v_5, v_6\rangle \supseteq 1$, so we again have a contradiction.

We conclude E is strongly closed in T with respect to G. As mentioned above $E \lhd G$ follows. The proof of the lemma and the theorem is complete.

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